A Remark on Normal Forms of Matrices

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INTRODUCTION

This remark answers the two problems raised in [2]. As in [4], we use the recent techniques of [3] and [5] of the representation theory of finite-dimensional algebras. It seems that these techniques provide methods of solution, as well as proper understanding, of such classification problems.

1. FIRST PROBLEM

The first problem of [2] asks for normal forms of $2m\times 2n$ complex matrices with respect to $\mathbb{H}$-similarity. Here, two complex $2m\times 2n$ matrices A,A' are said to be $\mathbb{H}$-*similar* if there exist formally quaternionic invertible (square) matrices P,Q such that $QA=A'P$, and a complex $2m\times 2n$ matrix P is called *formally quaternionic* if each block in its natural partition into 2×2 blocks has the form

$$\begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \qquad \text{with} \quad a,b\in\mathbb{C},$$

where $\bar{c}$ denotes the complex conjugate of $c\in\mathbb{C}$.

In order to solve this problem, one may just follow the general procedure presented in [4] and illustrated there on the classification problem of $2m\times 2n$ real matrices with respect to $\mathbb{C}$-similarity. There, the problem was reformulated as the classification of real linear maps between two complex vector spaces. Similarly, in the present problem, we are concerned with the

 0024-3795/80/020109+6$01.75

classification of complex linear maps ψ between two quaternionic vector spaces $V_{\mathbb{H}}$ and $W_{\mathbb{H}}$. It is easy to see that this is equivalent to the classification of pairs of linear maps between two quaternionic spaces. Indeed, the $\mathbb{C}$-linear maps

$$\psi: V_{\mathbb{H}}\otimes_{\mathbb{H}}\mathbb{H}_{\mathbb{C}}\to W_{\mathbb{H}}\otimes_{\mathbb{H}}\mathbb{H}_{\mathbb{C}}\approx \operatorname{Hom}_{\mathbb{H}}({}_{\mathbb{C}}\mathbb{H}_{\mathbb{H}}, W_{\mathbb{H}})$$

are in one-to-one correspondence with the $\mathbb{H}$-linear maps

$$\varphi: V_{\mathbb{H}}\otimes_{\mathbb{H}}\mathbb{H}_{\mathbb{C}}\otimes_{\mathbb{C}}\mathbb{H}_{\mathbb{H}}\to W_{\mathbb{H}},$$

and

$${}_{\mathbb{H}}\mathbb{H}_{\mathbb{C}}\otimes_{\mathbb{C}}\mathbb{H}_{\mathbb{H}} = {}_{\mathbb{H}}\mathbb{H}_{\mathbb{H}}\oplus {}_{\mathbb{H}}\mathbb{H}_{\mathbb{H}},$$

where the direct summends are generated by $1\otimes 1 - j\otimes j$ and $(1\otimes 1 + j\otimes j)i = 1\otimes i - j\otimes k$ (here $1,i,j,k$ is the standard basis of $\mathbb{H}_{\mathbb{R}}$).

Thus, as in [4], we just translate the classification of the indecomposable representation of the species $\mathbb{H}\xrightarrow{\mathbb{H}\oplus\mathbb{H}}\mathbb{H}$ into matrix language (choosing suitable bases). Using the terminology and results of [3] and [5], we know that the subcategory of the homogeneous representations is the product of a uniserial category with one simple object and the (also uniserial) category of all finite-dimensional modules over the polynomial algebra $\mathbb{H}[x]$, and we can obtain the simple objects of these categories from the Addendum of [3]. Thus we get the following

Theorem 1. *Every (nonzero) complex $2m\times 2n$ matrix is $\mathbb{H}$-similar to a zero-augmented product of matrices of the following types:*

(i) $2(p+1)\times 2p$ *matrices* $(p=1,2,\ldots)$

$$\begin{bmatrix} E_1 & & & & 0 \\ E_{-1} & E_1 & & & \\ & E_{-1} & E_1 & & \\ & & \ddots & \ddots & \\ 0 & & & E_{-1} & E_1 \\ & & & & E_{-1} \end{bmatrix},$$

(ii) *the corresponding transposed* $2p\times 2(p+1)$ *matrices* ($p=1,2,\ldots$), *and*
(iii) $2p\times 2p$ *matrices* ($p=1,2,\ldots$)

$$\begin{bmatrix} E_c & E_1 & & & \\ & E_c & E_1 & & \mathbf{0} \\ & & \ddots & \ddots & \\ \mathbf{0} & & & E_c & E_1 \\ & & & & E_c \end{bmatrix} \quad \textit{with} \quad c=a+bi,\ a\geqslant 0,$$

or

$$\begin{bmatrix} E & E_1 & & & \mathbf{0} \\ & E & E_1 & & \\ & & \ddots & \ddots & \\ \mathbf{0} & & & E & E_1 \\ & & & & E \end{bmatrix},$$

where

$$E_c=\begin{pmatrix} 1-c & 0 \\ 0 & 1+\bar{c} \end{pmatrix} \textit{ for complex } c \quad \textit{and} \quad E=\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}.$$

These matrices are $\mathbb{H}$*-indecomposable (i.e. not* $\mathbb{H}$*-similar to a proper direct product of two matrices), and in the decomposition of a complex* $2m\times 2n$ *matrix, they are determined (up to their order) uniquely.*

2. SECOND PROBLEM

The second problem asks for normal forms of $4m\times 4n$ real matrices with respect to $\mathbb{H}$-similarity. Here, two real $4m\times 4n$ real matrices A,A' are said to be $\mathbb{H}$-*similar* if there exist formally real-quaternionic invertible (square) matrices P,Q such that $QA=A'P$, and a real $4m\times 4n$ matrix P is called *formally real-quaternionic* if each block of its natural partition into 4×4

blocks has the form

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & d & -c \\ c & -d & a & b \\ d & c & -b & a \end{bmatrix} \quad \text{with} \quad a,b,c,d \in \mathbb{R}.$$

Similarly to the first problem, the $\mathbb{H}$-similarity classes of real $4m \times 4n$ matrices correspond to the isomorphism classes of real linear maps between two quaternionic vector spaces, and again, since ${}_{\mathbb{H}}\mathbb{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}_{\mathbb{H}} \approx \mathbb{H}^4$, they correspond to the isomorphism classes of representations of the species $\mathbb{H} \xrightarrow{\mathbb{H}^4} \mathbb{H}$. It is well known that this problem is "wild," so that one cannot expect a satisfactory normal form. For the benefit of the reader, we repeat here the argument.

Given any real $4m \times 4n$ matrix A, let

$$\mathrm{End}_{\mathbb{H}}(A) = \{(P,Q) \mid P,Q \text{ formally real-quaternionic matrices with } QA = AP\}$$

be its $\mathbb{H}$-*endomorphism ring* (the ring operations are componentwise). It is clear that for $\mathbb{H}$-similar real $4m \times 4n$ matrices A, A' the $\mathbb{R}$-algebras $\mathrm{End}_{\mathbb{H}}(A)$ and $\mathrm{End}_{\mathbb{H}}(A')$ are isomorphic.

THEOREM 2. *For any finite dimensional $\mathbb{R}$-algebra R, there exists a real $4m \times 4n$ matrix A with $\mathrm{End}_{\mathbb{H}}(A)$ isomorphic to R.*

Proof. Let R be generated by $r_1, \ldots, r_n$. We will consider the left multiplication by r_i as an element of $\mathrm{End}_{\mathbb{R}}(R)$, and denote it also by r_i. It is easy to check that the centralizer of the two $m \times m$ matrices

$$\alpha = \begin{bmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \mathbf{0} \\ & & 0 & 1 & & \\ & & & \ddots & \ddots & \\ \mathbf{0} & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 0 & & & & & \\ 1 & 0 & & & & \mathbf{0} \\ r_1 & 1 & 0 & & & \\ & \ddots & \ddots & \ddots & & \\ \mathbf{0} & & r_{n-1} & 1 & 0 & \\ & & & r_n & 1 & 0 \end{bmatrix}$$

in $\mathrm{End}_{\mathbb{R}}(X)$, with

$$X = \bigoplus_m R \quad \text{and} \quad m = n+2,$$

is just the set of R-multiples of the identity, and thus isomorphic to R (see [1]). Consider now $Y_{\mathbb{H}} = X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{H}_{\mathbb{H}}$. The centralizer of $1 \otimes i$ and $1 \otimes j$ (where i, j denote the corresponding left multiplications on $\mathbb{H}$) in $\mathrm{End}(Y_{\mathbb{H}})$ are the elements $\varphi \otimes 1$ with $\varphi \in \mathrm{End}(X_{\mathbb{R}})$; thus the centralizer of $1 \otimes i$, $1 \otimes j$ and $\alpha \otimes 1 + \beta \otimes i$ in $\mathrm{End}(Y_{\mathbb{H}})$ will be isomorphic to R. However, this centralizer is precisely the endomorphism ring $\mathrm{End}_{\mathbb{H}}(A)$ of the real $4m \times 4m$ matrix A corresponding to the representation

$$Y_{\mathbb{H}} \xrightarrow{\begin{array}{c} 1\otimes 1 \\ 1\otimes i \\ 1\otimes j \\ (\alpha\otimes 1)+(\beta\otimes i) \end{array}} Y_{\mathbb{H}}$$

of $\mathbb{H} \overset{\mathbb{H}^4}{\to} \mathbb{H}$. ∎

It follows from this theorem that a classification of real $4m \times 4n$ matrices with respect to $\mathbb{H}$-similarity is impossible: it would lead, at the same time, to a classification of all finite-dimensional $\mathbb{R}$-algebras.

3. CONCLUSION

Note that the first "open problem" is in fact, as we have shown above, a special case of the situation considered in the same paper [2]. It may perhaps be proper to emphasize two different objectives in dealing with classification problems: One is to find normal forms for a given problem; the other, usually easier objective is to show that two given problems have the same normal forms (modulo various discrete series of forms). The main theorem of [2] is a result of the second type (whereas the above solution of the first problem is of the first type). Let us remark that in such a situation, no simple normal form of matrices need exist at all—the classification of the similarity classes of matrices over a division ring seems to be a very difficult problem. On the other hand, the normal forms of matrices of discrete dimension type can always be listed [5], even in the "wild" situation of the second problem.

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Received 10 October 1978; revised May 1979